FUNCTIONS SATISFYING A WEIGHTED AVERAGE PROPERTY[1]

BY
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Introduction. It is well known that a harmonic function $f(x) = f(x_1, x_2, \cdots, x_n)$ defined in a given region (open connected set) R of the n-dimensional euclidean space E_n can be characterized by the following mean-value properties, namely,

$$f(x_0) = \frac{1}{V_n} \int_{B(x_0,r)} f(x)\, d\rho, \qquad x_0 \in R, \tag{0.1}$$

$$f(x_0) = \frac{1}{\Omega_n} \int_{S(x_0,r)} f(x)\, d\sigma, \tag{0.2}$$

where $B(x_0, r)$ and $S(x_0, r)$ denote any ball and its surface with x_0 for its center and radius r which lies in R; $d\rho$ and $d\sigma$ stand for the usual Lebesgue measures of B and S; V_n and Ω_n denote the total measures of B and S. We use the letters x and y to denote the n-dimensional vectors $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$.

Generalization of the above mean-value properties has been made in two directions, namely,

(I) in replacing the figures B and S by other figures, and

(II) in replacing the Lebesgue measure by other measures.

A number of authors have dealt with generalization (I). In the case of two dimensions, it has been proved by J. L. Walsh [11], E. F. Beckenbach and M. O. Reade [10], A. Friedman [7] and others that if the figures B and S are replaced, respectively, by a regular polygon of N sides and its boundary, then the functions having the mean-value properties (0.1) and (0.2) are precisely the harmonic polynomials of degree $\leqq N$ having zero Nth derivatives in each of the directions of the radii of the polygon.

Recently, in two very nice papers, L. Flatto [4] and A. Friedman and W. Littman [6] have generalized the above mean-value properties (0.1) and (0.2) simultaneously in the two directions (I) and (II). They were interested in characterizing the class of functions which satisfy the following mean-value properties:

$$f(x) = \int_K f(x + ty)\, d\mu(y), \tag{0.3}$$

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for all $x \in R$ and all $t \in (0, d(x, T))$, T being the boundary of R, where μ is a non-negative Borel measure in n-dimensional euclidean space E_n, having compact support K which lies in the unit sphere, and has total measure 1 (i.e., $\int_K d\mu = 1$).

They have shown that the mean-value property (0.3) is equivalent to the system of partial differential equations with constant coefficients,

$$\sum_{|\gamma| = j} A_\gamma D_\gamma f = 0 \qquad (0 \leqq j < \infty), \tag{0.4}$$

where the coefficients A_γ are the usual moments (with respect to the origin) of the measure μ.

In this paper we propose to generalize the simple mean-value property (0.1) by a more general mean-value property of the form

$$u(x_0) = \frac{\int_{B(x_0, r)} u(x)\,\omega(x)\,d\rho}{\int_{B(x_0, r)} \omega(x)\,d\rho}, \tag{0.5}$$

for each ball $B(x_0, r)$ whose closure lies in R, which we call the weighted average property (W.A.P.), where ω is a weight function (W.F.) defined in R.

DEFINITION. *ω is a W.F. in R* means that

(a) ω is a non-negative, real-valued function defined in R, and

(b) ω is locally summable in R, i.e., if $P \in R$ and $0 < r < d(P, T)$, T being the boundary of R, then the Lebesgue integral $\int_{B(P,r)} \omega\, d\rho$ over $B(P, r)$ exists and

$$\int_{B(P,r)} \omega\, d\rho > 0.$$

DEFINITION. A real-valued function u is said to satisfy the W.A.P. with respect to a W.F. ω in R provided $u \cdot \omega$ is locally summable in R and u satisfies the mean-value property (0.5) for each ball $B(x_0, r)$ whose closure lies in R.

DEFINITION.

$$R^* = \{(P, r)\colon P \in R \text{ and } 0 < r < d(P, T)\}.$$

Indeed, the mean-value property (0.5) is a generalization of the mean-value property (0.1) in the direction (II) mentioned at the beginning.

For suppose μ be a σ-finite measure on the σ-ring of Lebesgue measurable subsets of R such that

(i) μ is absolutely continuous with respect to the Lebesgue measure ρ, and

(ii) $\mu(B) > 0$ for each ball B lying in R.

By the Radon-Nikodym theorem there exists a W.F., ω, defined on R such that

$$\mu(E) = \int_E \omega\, d\rho \tag{0.6}$$

for every Lebesgue measurable subset E of R and the simple mean-value property (with respect to the measure μ):

$$(0.7) \qquad u(x_0) = \frac{\int_{B(x_0,r)} u\, d\mu}{\int_{B(x_0,r)} d\mu}, \qquad x_0 \in R,$$

for each ball $B(x_0,r)$ whose closure lies in R, is equivalent to the W.A.P. (0.5) with respect to this W.F. ω.

I. Suppose that R be a region in E_n and ω be a W.F. defined in R. Let $S(\omega,R)$ be the class of all functions satisfying W.A.P. with respect to ω in R.

Some basic facts about $S(\omega,R)$. It is clear from the definition of W.A.P. that

(0.8) $S(\omega,R)$ is nonempty (constant functions belong to $S(\omega,R)$),

(0.9) $S(\omega,R)$ is a linear space over the reals and $\dim S(\omega,R) \geqq 1$,

(0.10) if ω be a nonzero constant, then $S(\omega,R)$ is the class of all harmonic functions defined on R.

Continuity and differentiability of functions belonging to $S(\omega,R)$. In proving our first few theorems we will use the following well-known results (all functions are considered to be real-valued functions):

LEMMA 1. *If f be a locally summable function in R, then the integral $\int_{B(P,r)} f\, d\rho$ is continuous in P in R, i.e., if $P_0 \in R$ and $0 < r < d(P_0,T)$, then there exists a neighborhood $N(P_0)$ of the point P_0, lying in R, such that $\int_{B(P,r)} f\, d\rho$ is defined for each $P \in N(P_0)$, and*

$$\lim_{P \to P_0} \int_{B(P,r)} f\, d\rho = \int_{B(P_0,r)} f\, d\rho.$$

LEMMA 2. *If f be a continuous function defined in R, then the integral $\int_{B(P,r)} f\, d\rho$ is differentiable in P in R, i.e., if $P_0 \in R$ and $0 < r < d(P_0 T)$, then there exists a neighborhood $N(P_0)$ of the point P_0, lying in R, such that $F(P) = \int_{B(P,r)} f\, d\rho$ is defined for each point $P(x_1, x_2, \cdots, x_n)$ in $N(P_0)$, $\partial F/\partial x_i$ exists at P_0 and $\partial F/\partial x_i|_{P_0} = \int_{S(P_0,r)} f\, d(i)$, where $d(i) = dx_1 dx_2 \cdots dx_{i-1} dx_{i+1} \cdots dx_n$, $i = 1, 2, \cdots, n$.*

LEMMA 3. *If f be a continuous function defined in R, then each of the integrals*

$$\int_{S(P,r)} f\, d(i) = \int_{S(P,r)} f\, dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n, \qquad i = 1, 2, \cdots, n,$$

is continuous in P in R, i.e., if $P_0 \in R$ and $0 < r < d(P_0,T)$ then there exists a neighborhood $N(P_0)$ of the point P_0, lying in R, such that $\int_{S(P,r)} f\, d(i)$ is defined for each point P in $N(P_0)$ and

$$\lim_{P\to P_0}\int_{S(P,r)} f\,d(i) = \int_{S(P_0,r)} f\,d(i), \qquad i = 1,2,\cdots,n.$$

The above three results follow, respectively, from
(i) absolute continuity of the Lebesgue integral,
(ii) the continuity of the integrand f, and
(iii) the uniform continuity of f on compact subsets of R.

THEOREM 1. *If $u \in S(\omega, R)$, then u is continuous in R.*

Proof. Let $P_0 \in R$ and $0 < r < d(P_0, T)$; then by Lemma 1, there exists a neighborhood $N(P_0)$ of the point P_0, lying in R, such that each of the integrals $\int_{B(P,r)} u\omega\, d\rho$ and $\int_{B(P,r)} \omega\, d\rho$ exists for $P \in N(P_0)$ and

$$\lim_{P\to P_0}\int_{B(P,r)} u\omega\, d\rho = \int_{B(P_0,r)} u\omega\, d\rho \text{ and } \lim_{P\to P_0}\int_{B(P,r)} \omega\, d\rho = \int_{B(P_0,r)} \omega\, d\rho \quad (\neq 0).$$

Hence

$$\lim_{P\to P_0} u(P) = \lim_{P\to P_0} \frac{\int_{B(P,r)} u\omega\, d\rho}{\int_{B(P,r)} \omega\, d\rho} = \frac{\int_{B(P_0,r)} u\omega\, d\rho}{\int_{B(P_0,r)} \omega\, d\rho} = u(P_0).$$

This proves the continuity of u.

THEOREM 2. (i) *If the W.F. ω belongs to class $C^m(R)$, where m is a non-negative integer and $u \in S(\omega, R)$, then $u \in C^{m+1}(R)$.*

(ii) *If ω be infinitely differentiable in R and $u \in S(\omega,R)$, then u is infinitely differentiable in R.*

(iii) *If ω be analytic in R and $u \in S(\omega, R)$, then u is analytic in R.*

REMARK. Here we will prove only part (i) of Theorem 2 for the cases $m = 0$ and 1, and avoid giving a direct proof for the remaining parts of the theorem which is not difficult but rather too long.

The case $m > 1$, parts (ii) and (iii) of Theorem 2 will follow as corollaries to Theorem 4.

Proof.

Case $m = 0$, i.e., ω is a continuous W.F. By Lemma 2 and Theorem 1, each of the integrals $\int_{B(P,r)} u\omega\, d\rho$ and $\int_{B(P,r)} \omega\, d\rho$ is differentiable in R. Since

$$u(P) = \frac{\int_{B(P,r)} u\omega\, d\rho}{\int_{B(P,r)} \omega\, d\rho} \qquad \text{for all } (P,r) \in R^*,$$

it follows that u is differentiable in R and each of the derivatives $\partial u/\partial x_i = u_{x_i}$, $i = 1,2,\cdots,n$, is given by

$$u_{x_i}(P) = \frac{\int_{B(P,r)} \omega\, d\rho \int_{S(P,r)} u\omega\, d(i) - \int_{B(P,r)} u\omega\, d\rho \int_{S(P,r)} \omega\, d(i)}{\left\{\int_{B(P,r)} \omega\, d\rho\right\}^2}$$

or

$$u_{x_i}(P) = \frac{\int_{S(P,r)} u\omega\, d(i) - u(P)\int_{S(P,r)} \omega\, d(i)}{\int_{B(P,r)} \omega\, d\rho} \quad \text{for all } (P,r) \in R^*. \tag{1.1}$$

By Lemma 1 and Lemma 3 each of the integrals involved in (1.1) is continuous in R, which means that $u \in C'(R)$.

Case $m = 1$. Suppose that $\omega \in C'(R)$. Then by the previous case $u\omega \in C'(R)$ and, using the divergence theorem, we can write (1.1) as

$$u_{x_i}(P) = \frac{\int_{B(P,r)} (u\omega)_{x_i}\, d\rho - u(P)\int_{B(P,r)} \omega_{x_i}\, d\rho}{\int_{B(P,r)} \omega\, d\rho}, \qquad i = 1,2,\cdots,n. \tag{1.2}$$

Since each of the integrals involved in (1.2) is differentiable in R, it follows that each of the partial derivatives u_{x_i}, $i = 1,2,\cdots,n$, is differentiable in R and, furthermore, at each point P in R, $u_{x_ix_j}(P)$ $(i = 1,2,\cdots,n;\ j = 1,2,\cdots,n)$ is given by

$$u_{x_ix_j}(P) = \frac{\int_{S(P,r)} (u\omega)_{x_i}\, d(j) - u_{x_i}(P)\int_{B(P,r)} \omega_{x_j}\, d\rho - u_{x_j}(P)\int_{B(P,r)} \omega_{x_i}\, d\rho - u(P)\int_{S(P,r)} \omega_{x_i}\, d(j)}{\int_{B(P,r)} \omega\, d\rho} \tag{1.3}$$

for each r satisfying $0 < r < d(P,T)$. Each of the integrals involved in (1.3) is continuous in R which means that $u \in C^2(R)$.

Circumferential Weighted Average Property.

THEOREM 3. *Let ω be a continuous W.F. defined in R. A necessary and sufficient condition that $u \in S(\omega, R)$ is that u is continuous in R and satisfies the following circumferential W.A.P. with respect to ω in R:*

$$u(P)\int_{S(P,r)} \omega\, d\sigma = \int_{S(P,r)} u\omega\, d\sigma \quad \textit{for all } (P,r) \in R^*. \tag{1.4}$$

Proof. It is enough to give the proof for three dimensions. The proof for the general case is quite similar.

Sufficiency. Suppose u be a continuous function defined in R, such that u satisfies the circumferential W.A.P. (1.4). Using spherical coordinates (r,θ,ϕ), (1.4) can be written as

$$u(P)\int_0^{\pi}\int_0^{2\pi}\omega r^2\sin\phi\, d\theta d\phi = \int_0^{\pi}\int_0^{2\pi} u\omega r^2 \sin\phi\, d\theta d\phi. \tag{1.5}$$

Integrating both sides of (1.5) with respect to r from 0 to r we have, since $u(P)$ is independent of r,

$$u(P)\int_0^{r}\int_0^{\pi}\int_0^{2\pi}\omega r^2\sin\phi\, d\theta d\phi dr = \int_0^{r}\int_0^{\pi}\int_0^{2} u\omega r^2 \sin\phi\, d\theta d\phi dr$$

or

$$u(P)\int_{B(P,r)}\omega\, d\rho = \int_{B(P,r)} u\omega\, d\rho \quad \text{for each } (P,r)\in R^*.$$

Hence $u \in S(\omega,R)$.

Necessity. Suppose that $u\in S(\omega,R)$. Then, by Theorem 1, u is continuous in R, and

$$u(P)\int_{B(P,r)}\omega\, d\rho = \int_{B(P,r)} u\omega\, d\rho \quad \text{for each } (P,r)\in R^*. \tag{1.6}$$

Using spherical coordinates, (1.6) can be written as

$$u(P)\int_0^{r}\int_0^{\pi}\int_0^{2\pi}\omega r^2\sin\phi\, d\theta d\phi dr = \int_0^{r}\int_0^{\pi}\int_0^{2\pi} u\omega r^2 \sin\phi\, d\theta d\phi dr. \tag{1.7}$$

Differentiating both sides of (1.7) with respect to r (which is possible under the hypothesis), we get

$$u(P)\int_0^{\pi}\int_0^{2\pi}\omega r^2\sin\phi\, d\theta d\phi = \int_0^{\pi}\int_0^{2\pi} u\omega r^2 \sin\phi\, d\theta d\phi$$

or

$$u(P)\int_{S(P,r)}\omega\, d\sigma = \int_{S(P,r)} u\omega\, d\sigma \quad \text{for each } (P,r)\in R^*.$$

This completes the proof.

Differential equation.

THEOREM 4. *If the W.F. $\omega\in C'(R)$ and $u\in S(\omega,R)$, then u satisfies the second order elliptic differential equation*

$$\omega\Delta u + 2\sum_{i=1}^{n}\frac{\partial\omega}{\partial x_i}\cdot\frac{\partial u}{\partial x_i} = 0 \quad \textit{in } R. \tag{1.8}$$

Proof. By Theorem 2, $u \in C^2(R)$. Let $P \in R$ and $0 < r < d(P,T)$; then by Theorem 3,

$$u(P)\int_{S(P,r)} \omega\, d\sigma = \int_{S(P,r)} u\omega\, d\sigma. \tag{1.9}$$

Differentiating with respect to r we get

$$u(P)\int_{S(P,r)} \frac{\partial \omega}{\partial r}\, d\sigma = \int_{S(P,r)} \frac{\partial (u\omega)}{\partial r}\, d\sigma$$

or

$$u(P)\int_{S(P,r)} \frac{\partial \omega}{\partial n}\, d\sigma = \int_{S(P,r)} \frac{\partial (u\omega)}{\partial n}\, d\sigma, \tag{1.10}$$

where $\partial/\partial n$ refers to the directional derivative in the direction of the outward drawn normal to the surface $S(P,r)$.

Again, by (1.3), $\partial^2 u/\partial x_i^2|_P = u_{x_i x_i}(P)$ is given by

$$u_{x_i x_i}(P)\int_{B(P,r)} \omega\, d\rho + 2u_{x_i}(P)\int_{B(P,r)} \omega_{x_i}\, d\rho = \int_{S(P,r)} (u\omega)_{x_i} d(i) - u(P)\int_{S(P,r)} \omega_{x_i}\, d(i)$$

for $i = 1,2,\cdots,n$.

Hence

$$\begin{aligned}
&\int_{B(P,r)} \omega\, d\rho \sum_{i=1}^{n} u_{x_i x_i}(P) + 2\sum_{i=1}^{n} u_{x_i}(P)\int_{B(P,r)} \omega_{x_i}\, d\rho \\
&= \int_{S(P,r)} \sum_{i=1}^{n} (u\omega)_{x_i} d(i) - u(P)\int_{S(P,r)} \sum_{i=1}^{n} \omega_{x_i} d(i) \\
&= \int_{S(P,r)} \frac{\partial (u\omega)}{\partial n}\, d\sigma - u(P)\int_{S(P,r)} \frac{\partial \omega}{\partial n}\, d\sigma \\
&= 0,
\end{aligned} \tag{1.11}$$

by (1.10), or

$$\Delta u(P)\left(\lim_{r\to +0} \frac{1}{V_n}\int_{B(P,r)} \omega\, d\rho\right) + 2\sum_{i=1}^{n} u_{x_i}(P)\left(\lim_{r\to +0} \frac{1}{V_n}\int_0 \omega_{x_i}\, d\rho\right) = 0$$

or

$$\Delta u(P)\omega(P) + 2\sum_{i=1}^{n} u_{x_i}(P)\omega_{x_i}(P) = 0$$

or

$$\omega(P)\Delta u(P) + 2\sum_{i=1}^{n} \omega_{x_i}(P) u_{x_i}(P) = 0, \tag{1.12}$$

where V_n is the measure of $B(P,r)$ and $\Delta u(P)$ is the Laplacian of u evaluated at the point P. Since P is an arbitrary point in R, it follows that u satisfies the differential equation

$$\omega\Delta u + 2\sum_{i=1}^{n} \frac{\partial\omega}{\partial x_i} \cdot \frac{\partial u}{\partial x_i} = 0 \quad \text{in } R.$$

The above equation is clearly of elliptic type.

REMARK. Part (i) for the case $m > 1$, part (ii) and part (iii) of Theorem 2 now follow from the corresponding and well-known properties of the solutious of the differential equation (1.8) [2].

REMARK. If ω is a nonzero constant, then the differential equation (1.8) reduces to the Laplacian equation, $\Delta u = 0$, as is expected, because $S(\omega, R)$ in this case is the class of all harmonic functions defined over R.

W.A.P. and solutions of differential equation (1.8). Theorem 4 states that if the W.F. $\omega \in C'(R)$, then each member of $S(\omega, R)$ satisfies the differential equation (1.8). A question therefore naturally arises "whether each solution of (1.8) which belongs to class $C^2(R)$ is necessarily a member of $S(\omega, R)$?" That the answer is no is provided by the following counterexample:

(1.13) Let R be the whole plane E_2 and consider the W.F. $\omega(x, y) = (x + y)^2$, $\omega \in C'(R)$. Each of the following two functions belongs to $C^2(R)$ and is a solution of the equation (1.8) in E_2: $u(x, y) = x - y$, $v(x, y) = x^2 - 3xy + y^2$. It can be verified directly by calculating the integrals involved that $u \in S(\omega, R)$ but $v \notin S(\omega, R)$.

Weight functions which are solutions of $\Delta\omega + \lambda\omega = 0$.

THEOREM 5. *Let ω be a W.F. defined in R and λ a real number such that $\omega \in C^2(R)$ and is a solution of the differential equation*

$$\Delta F + \lambda F = 0 \quad \textit{in } R. \tag{2.1}$$

Then a necessary and sufficient condition that $u \in S(\omega, R)$ is that

(i) $u \in C^2(R)$; *and*

(ii) *u satisfies the differential equation*

$$\omega\Delta u + 2\sum_{i=1}^{n} u_{x_i}\omega_{x_i} = 0 \quad \textit{in } R. \tag{2.2}$$

Proof. We will use the following established result (for proof see [2, p. 289]):

LEMMA 4. *If $f \in C^2(R)$ and is a solution of* (2.1) *in R then f satisfies the mean value property*:

$$\int_{S(P,r)} f\,d\sigma = f(P) \cdot K(r) \quad \textit{for all } (P, r) \in R^*, \tag{2.3}$$

where

$$K(r) = \Omega_n \frac{\Gamma\left(\frac{n}{2}\right) J_{(n-2)/2}(r\sqrt{\lambda})}{\left(\frac{r\sqrt{\lambda}}{2}\right)^{(n-2)/2}} \qquad \textit{for } \lambda > 0,$$

(1)

$$= \Omega_n \frac{\Gamma\left(\frac{n}{2}\right) I_{(n-2)/2}(r\sqrt{-\lambda})}{\left(\frac{r\sqrt{-\lambda}}{2}\right)^{(n-2)/2}} \qquad \textit{for } \lambda < 0,$$

$$= \Omega_n \qquad \textit{for } \lambda = 0,$$

(2) Ω_n = *measure of* $S(P,r)$,

(3) $J_{(n-2)/2}$ *and* $I_{(n-2)/2}$ *are, respectively, the Bessel and modified Bessel functions of order* $(n-2)/2$, *and* Γ *stands for the usual* Γ*-function.*

Necessity. The necessity of the theorem is a direct consequence of Theorem 4.

Sufficiency. Suppose that u satisfies the conditions (i) and (ii), then the function $F = u\omega$ belongs to class $C^2(R)$ and is a solution of (2.1). Hence, by Lemma 4,

$$\int_{S(P,r)} \omega \, d\sigma = \omega(P) \cdot K(r), \tag{2.4}$$

$$\int_{S(P,r)} u\omega \, d\sigma = u(P) \cdot \omega(P) \cdot K(r) \tag{2.5}$$

for all $(P,r) \in R^*$.

Again,

$$\omega(P) > 0 \quad \text{for each } P \text{ in } R. \tag{2.6}$$

For $\omega(P) = 0$ implies by (2.4) that $\int_{S(P,r)} \omega \, d\sigma = 0$ for each r satisfying $0 < r < d(P,T)$ which again implies that $\omega(Q) = 0$ for all Q in $S(P,r)$. This means that $\omega(Q) = 0$ for all Q in $B(P,r)$. Hence $\int_{B(P,r)} \omega \, d\rho = 0$ which contradicts the fact that ω is a W.F.

Therefore $\int_{S(P,r)} \omega \, d\sigma > 0$ for all $(P,r) \in R^*$.

Hence $K(r) \neq 0$. We therefore have

$$\frac{\int_{S(P,r)} u\omega \, d\sigma}{\int_{S(P,r)} \omega \, d\sigma} = u(P) \qquad \text{for all } (P,r) \in R^*.$$

This means, by Theorem 3, that $u \in S(\omega, R)$.

COROLLARY 1. *Let* ω *be a W.F. defined in* R *and* λ *a real number such that* $\omega \in C^2(R)$ *and is a solution of the differential equation*

$$\Delta F + \lambda F = 0, \quad \textit{in } R. \tag{2.7}$$

Then the following are true:

(a) *If* $f \in C^2(R)$, *and is a solution of* (2.7), *then* $f/\omega \in S(\omega, R)$;

(b) *If* $u \in S(\omega, R)$, *then* $u\omega \in C^2(R)$ *and is a solution of* (2.7).

Proof. *Part* (a). By (2.6), ω cannot vanish in R, hence, $f/\omega \in C^2(R)$, and it can be shown by direct calculation of the derivatives that f/ω satisfies the differential equation (1.8). Therefore, by Theorem 5, $f/\omega \in S(\omega, R)$.

Part (b). Again $u \in S(\omega, R)$ implies, by Theorem 2, that $u\omega \in C^2(R)$ and, writing $G = u\omega$, it can be shown by direct calculation of the Laplacians that the following relation is true:

$$\omega \Delta G = G \Delta \omega \quad \text{in } R. \tag{2.8}$$

Since ω satisfies (2.7), it follows that $\Delta G + \lambda G = 0$ in R.

Dimension of $S(\omega, R)$.

COROLLARY 2. *Let* ω *be a W.F. defined in a region* R *of* E_n $(n > 1)$ *and* λ *a real number such that* $\omega \in C^2(R)$, *and is a solution of the differential equation*

$$\Delta F + \lambda F = 0 \quad \text{in } R. \tag{2.9}$$

Then $S(\omega, R)$ *is infinite dimensional.*

Proof. Since the solution space of the differential equation (2.9) is infinite dimensional, it follows easily from part (a) of Corollary 1 that $S(\omega, R)$ is also infinite dimensional.

REMARK. In E_1, the following can be proved easily: If the W.F. $\omega \in C^1(R)$ then

(i) $1 \leqq \dim S(\omega, R) \leqq 2$,

(ii) $\dim S(\omega, R) = 2$ if and only if $\omega \in C^2(R)$ and there is a real number λ such that ω is a solution of $d^2\omega/dx^2 + \lambda(d\omega/dx) = 0$ in R.

Example of a finite-dimensional $S(\omega, R)$ *in* E_2. Let R be the entire plane E_2 and ω the W.F.:

$$\omega(x, y) = (x + y)^2, \qquad (x, y) \in E_2.$$

It can be proved easily that $\dim S(\omega, R) = 2$ and functions belonging to $S(\omega, R)$ are the constants and linear functions of the form $C_1(x - y) + C_2$, where C_1 and C_2 are arbitrary real constants.

Application of divergence theorem.

COROLLARY. *If the W.F.* $\omega \in C^1(R)$, *and* $u \in S(\omega, R)$, *then*

$$\int_{S(P,r)} \omega^2 \frac{\partial u}{\partial n} \, d\sigma = 0 \qquad \textit{for all } (P, r) \in R^*, \tag{2.10}$$

where $\partial/\partial n$ *refers to directional derivative in the direction of the outward drawn normal to the surface* $S(P, r)$.

Proof. By Theorem 4, u satisfies the differential equation

$$\omega\Delta u + 2\sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\cdot\frac{\partial \omega}{\partial x_i} = 0 \quad \text{in } R,$$

which can be written as

$$\omega^2\Delta u + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\cdot\frac{\partial(\omega^2)}{\partial x_i} = 0 \quad \text{in } R.$$

Therefore, using the divergence theorem, we get

$$0 = \int_{B(P,r)}\left[\omega^2\Delta u + \sum_{i=1}^{n}\frac{\partial u}{\partial x_i^2}\cdot\frac{\partial(\omega^2)}{\partial x_i}\right]d\rho = \int_{S(P,r)}\omega^2\frac{\partial u}{\partial n}d\sigma \text{ for all } (P,r)\in R^*.$$

REMARK. If ω is a nonzero constant, then $S(\omega, R)$ is the class of all harmonic functions defined in R and relation (2.10) reduces to $\int_{S(P,r)}(\partial u/\partial n)\,d\sigma = 0$, which is a well-known property of harmonic functions. Again, vanishing of the integral $\int_{S(P,r)}(\partial u/\partial n)\,d\sigma$ at each point P in R and for all r satisfying $0 < r < d(P,T)$ implies that u is harmonic in R.

But if $\omega\in C^1(R)$ and $u\in C^2(R)$ such that

$$\int_{S(P,r)}\omega^2\frac{\partial u}{\partial n} = 0 \quad \text{for each } (P,r)\in R^*,$$

then it is not necessarily true that $u\in S(\omega,R)$. A counterexample is provided by the example (1.13); for the function $v(x,y) = x^2 - xy + y^2$ satisfies the differential equation $\omega\Delta u + 2(u_x\omega_x + u_y\omega_y) = 0$, where $\omega(x,y) = (x+y)^2$, and hence, satisfies (2.10), but v does not belong to $S(\omega,R)$.

COROLLARY 4. *If the W.F. $\omega\in C^2(R)$, and each of u and v belongs to $S(\omega,R)$, then*

$$\int_{S(P,r)}\omega\left\{u\frac{\partial(v\omega)}{\partial n} - v\frac{\partial(u\omega)}{\partial n}\right\}d\sigma = 0 \quad \textit{for all } (P,r)\in R^*. \tag{2.11}$$

Proof. It can be shown by direct calculations of the Laplacians that the following relations are true in R:

$$\begin{aligned} \omega\Delta(u\omega) &= u\omega\Delta\omega, \\ \omega\Delta(v\omega) &= v\omega\Delta\omega, \quad \text{therefore} \\ u\omega\Delta(v\omega) - v\omega\Delta(u\omega) &= 0 \quad \text{in } R. \end{aligned} \tag{2.12}$$

Using Green's second identity, we have

$$\begin{aligned} 0 &= \int_{B(P,r)}\omega\{u\Delta(v\omega) - v\Delta(u\omega)\}\,d\rho \\ &= \int_{S(P,r)}\omega\left\{u\frac{\partial(v\omega)}{\partial n} - v\frac{\partial(u\omega)}{\partial n}\right\}d\sigma \quad \text{for all } (P,r)\in R^*. \end{aligned}$$

REMARK. Corollaries 3 and 4 remain true if the boundary $S(P,r)$ of the ball $B(P,r)$ is replaced by the boundary of any subregion, whose closure lies in R, to which the divergence theorem applies.

II. *Maximum (minimum) principle.* The maximum principle for harmonic functions appears in two forms: the "weak principle" asserts that a harmonic function defined in the closure of a bounded region assumes its maximum on the boundary; the "strong principle" states that assumption of a maximum at an interior point implies the constancy of the function. In this section we shall state and prove a strong maximum (minimum) principle for functions satisfying W.A.P. (the weak principle is an immediate consequence).

Strong maximum (minimum) principle.

THEOREM 6. *If ω be a W.F. defined in a region R and $u \in S(\omega, R)$, then u cannot assume a maximum (minimum) at a point in R unless u is constant in R.*

Proof. Suppose that u assumes a maximum M at a point P_0 in R. We can get a ball $B(P_0,r)$, $0<r<d(P_0,T)$, such that $u(Q) \leqq M$ for all Q in $B(P_0,r)$. If possible, let P be a point in $B(P_0,r)$ such that $u(P)<M$. By the continuity of u we can find another ball $\bar{B}(P,r')$ such that $\bar{B}(P,r') \subset B(P_0,r)$ and $u(Q)<M$ for all Q in $\bar{B}(P,r')$. Using the mean-value theorem we get the following contradiction:

$$M = u(P_0) = \frac{\int_{B(P_0,r)} u\omega\, d\rho}{\int_{B(P_0,r)} \omega\, d\rho} = \frac{\int_{B(P_0,r)-B(P,r')} u\omega\, d\rho + \int_{B(P,r')} u\omega\, d\rho}{\int_{B(P_0,r)-B(P,r')} \omega\, d\rho + \int_{B(P,r')} \omega\, d\rho}$$

$$= \frac{u(P_1)\int_{B(P_0,r)-B(P,r')} \omega\, d\rho + u(P_2)\int_{B(P,r')} \omega\, d\rho}{\int_{B(P_0,r)-B(P,r')} \omega\, d\rho + \int_{B(P,r')} \omega\, d\rho} < M,$$

where $u(P_1) \leqq M$, $u(P_2) < M$.

Hence $u(P) = M$ for all P in $B(P_0,r)$. This means that the subset A of R where u assumes the maximum M ($P_0 \in A$) is open relative to R. By the continuity of u, A is also closed relative to R. Since R is connected, it follows that $A = R$. In other words u is constant in R. (The proof for the case of the minimum is exactly similar.)

REMARK. Many properties of harmonic functions which are direct consequences of these maximum (minimum) principles can also be extended to the class of functions satisfying W.A.P.

Dirichlet Problem. By the Dirichlet Problem for functions satisfying W.A.P., we mean the following:

(3.1) Let R be a bounded region with boundary T and ω a W.F. defined in R. If f is a continuous function defined on T, then the Dirichlet Problem for functions satisfying W.A.P. with respect to ω in R is that of determining a function u which

(1) is continuous on $\bar{R}$,

(2) belongs to $S(\omega, R)$,

(3) is identical with f on T.

Unicity of the solution of the Dirichlet Problem (3.1). It follows immediately from the maximum (minimum) principles that there can be at most one such function.

Existence of the solution of Dirichlet Problem (3.1). It is quite obvious that we cannot expect a solution of the Dirichlet Problem (3.1), for an arbitrary continuous boundary function f, if $S(\omega, R)$ is finite dimensional.

We know that, if the W.F. ω satisfies the hypothesis of Theorem 5, then $S(\omega, R)$ is infinite dimensional (Corollary 2), and every solution of the differential equation

$$\omega \Delta u + 2 \sum_{i=1}^{n} u_{x_i} \omega_{x_i} = 0, \tag{3.2}$$

which belongs to class $C^2(R)$, is a member of $S(\omega, R)$. But it is also well known that the solution of the Dirichlet Problem for the differential equation (3.2) exists (with the usual smoothness condition on the boundary T of the bounded region R; see [2, Chapter IV]). Hence we get the following theorem.

THEOREM 7. *Let ω be a W.F. defined in a bounded region R with smooth boundary T and λ a real number such that $\omega \in C^2(R)$ and is a solution of the differential equation*

$$\Delta F + \lambda F = 0 \quad \text{in } R. \tag{3.3}$$

Then the Dirichlet Problem for functions satisfying W.A.P. with respect to ω in R has a unique solution.

Properties of harmonic weight functions.

POISSON'S INTEGRAL FORMULA. It is known that for harmonic functions, the solution of the Dirichlet Problem for spheres is given by Poisson's integral formula, namely:

If f be a given continuous function defined on $S(0, r)$, where 0 is the origin then the function u,

$$u(P) = \frac{1}{\sigma_n r} \int_{S(0,r)} \frac{r^2 - |P|^2}{|P - Q|^n} f(Q)\, d\sigma_n(Q), \qquad P \in B(0, r), \tag{3.4}$$

is harmonic in $B(0, r)$ and $\lim_{P \to Q} u(P) = f(Q)$ for P in $B(0, r)$ and Q in $S(0, r)$, where $|P-Q|$ and $|P|$ represent the distance of P from Q and the origin, respectively, and σ_n is the n-dimensional surface area of the unit sphere.

A similar integral formula can be obtained for functions satisfying W.A.P. with respect to a harmonic W.F. (Theorem 7 guarantees the existence of the solution of the Dirichlet Problem for harmonic weight functions).

THEOREM 8. *Suppose* ω *be a W.F. harmonic in* $B(0,r)$ *and continuous on* $\bar{B}(0,r)$ *such that* $\omega(Q)>0$ *for* $Q\in S(0,r)$. *Let* f *be a given continuous function defined on* $S(0,r)$. *Then the function* u,

$$u(P)=\frac{1}{\sigma_n r}\int_{S(0,r)}\frac{r^2-|P|^2}{|P-Q|^n}\,\frac{\omega(Q)}{\omega(P)}\,f(Q)\,d\sigma_n(Q),\qquad P\in B(0,r), \tag{3.5}$$

satisfies W.A.P. with respect to ω *in* $B(0,r)$, *and*

$$\lim_{P\to Q} u(P) = f(Q) \quad \text{for } P\in B(0,r) \text{ and } Q\in S(0,r).$$

Proof. Since ω satisfies $\Delta\omega=0$ in $B(0,r)$, it follows from Corollary 1 that $h/\omega\in S(\omega,B(0,r))$ for every function h harmonic in $B(0,r)$; therefore, by (3.4) the function h,

$$h(P)=\frac{1}{\sigma_n r}\int_{S(0,r)}\frac{r^2-|P|^2}{|P-Q|^n}\,\omega(Q)f(Q)\,d\sigma_n(Q),\qquad P\in B(0,r),$$

is harmonic in $B(0,r)$ and $\lim_{P\to Q}h(P)=f(Q)\omega(Q)$ for P in $B(0,r)$ and Q in $S(0,r)$

Hence the function u,

$$u(P)=\frac{h(P)}{\omega(P)}=\frac{1}{\sigma_n r}\int_{S(0,r)}\frac{r^2-|P|^2}{|P-Q|^n}\,\frac{\omega(Q)}{\omega(P)}\,f(Q)\,d\sigma_n(Q),\quad P\in B(0,r),$$

satisfies W.A.P. with respect to ω in $B(0,r)$ and

$$\lim_{P\to Q} u(P)=\lim_{P\to Q}\frac{h(P)}{\omega(P)}=f(Q) \quad \text{for } P\in B(0,r) \text{ and } Q\in S(0,r).$$

Extension of Epstein's theorem. The following theorem has been proved by Professor Bernard Epstein [3].

"THEOREM. *Let* R *be a simply connected plane domain of finite area and* P *a point of* R *such that, for every function* u *harmonic in* R *and integrable over* R, *the mean value of* u *over the area of* R *equals* $u(P)$. *Then* R *is a disc and* P *is its center.*"

REMARK. Because of the close relationship between the class of harmonic functions and the class of functions satisfying W.A.P. with respect to a harmonic weight function, the above theorem can also be extended to the latter class of functions. As the proof is exactly similar to that of Epstein's theorem, we will give here only a brief outline of the proof.

THEOREM 9. *Let R be a simply connected plane region and ω a harmonic W.F. defined in R and integrable over R. Let P be a point in R such that, for every function u belonging to $S(\omega,R)$ and $u\omega$ integrable over R,*

$$u(P) = \frac{\int_R u\omega\, dxdy}{\int_R \omega\, dxdy}.$$

Then R is a disc and P is its center.

Proof. Let h be a harmonic function in R and integrable over R. By Corollary 1, h/ω belongs to $S(\omega,R)$ and $(h/\omega)\omega = h$ is integrable over R, hence

$$\frac{h(P)}{\omega(P)} = \frac{\int_R \left(\frac{h}{\omega}\right)\omega\, dxdy}{\int_R \omega\, dxdy} = \frac{\int_R h\, dxdy}{\int_R \omega\, dxdy}$$

or

$$h(P) = \int_R hA^{-1}\, dxdy, \tag{3.6}$$

where

$$A = \frac{1}{\omega(P)}\int_R \omega\, dxdy.$$

Now let $f(z)$ be any function analytic in R with finite quadratic integral $\int_R |f(z)|^2\, dxdy$; by the Schwarz inequality, the real and imaginary parts of $f(z)$ are each (absolutely) integrable over R, and so by (3.6) we have

$$f(P) = \int_R f(z)A^{-1}\, dxdy; \tag{3.7}$$

on the other hand, for every analytic quadratically integrable function, the equality

$$f(P) = \int_R f(z)K(z,P^*)\, dxdy \tag{3.8}$$

holds, where $K(z,P)$ denotes the "reproducing kernel" [1] of R with parameter-point P. Since $K(z,P)$ is uniquely determined by its reproducing property, it follows that

$$K(z,P) = A^{-1} \tag{3.9}$$

and hence that

$$\int_P^z K(\xi, P)\, d\xi = A^{-1}(z-P). \tag{3.10}$$

On the other hand, it is known that the function defined by the left side of (3.10) maps R into a disc with center at the origin. (It is at this point in the argument that the simple connectivity of R is employed.) It is now evident from the right side of (3.10) that R is itself a disc, with center at P.

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